A Freudenthal-Tits Supermagic Square

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Freudenthal-Tits Magic Square

A supermagic rectangle

A supermagic square

Some conclusions

Exceptional Lie algebras

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$$G_2, \quad F_4, \quad E_6, \quad E_7, \quad E_8$$

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$G_2 = \mathfrak{der}\,\mathbb{O}$ (Cartan 1914)

$F_4 = \mathfrak{der}\,H_3(\mathbb{O})$

$E_6 = \mathfrak{str}_0\,H_3(\mathbb{O})$

(Chevalley-Schafer 1950)

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then

$$\mathcal{T}(C, J) = \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J$$

is a Lie algebra (char $\neq 2, 3$) under a suitable Lie bracket:

$$[a \otimes x, b \otimes y] = \frac{1}{3} tr(xy) D_{a,b} + \Big([a, b] \otimes \big(xy - \frac{1}{3} tr(xy) 1\big)\Big) + 2t(ab) d_{x,y}.$$

Freudenthal-Tits Magic Square

$\mathcal{T}(C, J)$	$H_3(k)$	$H_3(k \times k)$	$H_3(\mathrm{Mat}_2(k))$	$H_3(C(k))$
k	A_1	A_2	C_3	F_4
$k \times k$	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathrm{Mat}_2(k)$	C_3	A_5	D_6	E_7
$C(k)$	F_4	E_6	E_7	E_8

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$$J = H_3(C') \simeq k^3 \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right),$$

$$J_0 \simeq k^2 \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right),$$

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$$\begin{aligned}
\mathcal{T}(C, J) &= \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J \\
&\simeq \mathfrak{der}\, C \oplus (C_0 \otimes k^2) \oplus \left(\oplus_{i=0}^{2} C_0 \otimes \iota_i(C')\right) \oplus \left(\mathfrak{tri}(C') \oplus (\oplus_{i=0}^{2} \iota_i(C'))\right) \\
&\simeq \left(\mathfrak{tri}(C) \oplus \mathfrak{tri}(C')\right) \oplus \left(\oplus_{i=0}^{2} \iota_i(C \otimes C')\right)
\end{aligned}$$

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$$\mathfrak{tri}(C) = \{(d_0, d_1, d_2) \in \mathfrak{so}(C)^3 : \overline{d_0(\overline{xy})} = d_2(x)y + x d_1(y)\ \forall x, y \in C\}$$

is the triality Lie algebra of C.

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is given by:

- $\mathfrak{tri}(C) \oplus \mathfrak{tri}(C')$ is a Lie subalgebra of $\mathfrak{g}(C, C')$,
- $[(d_0, d_1, d_2), \iota_i(x \otimes x')] = \iota_i\big(d_i(x) \otimes x'\big)$,
- $[(d_0', d_1', d_2'), \iota_i(x \otimes x')] = \iota_i\big(x \otimes d_i'(x')\big)$,
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = \iota_{i+2}\big((\bar{x}\bar{y}) \otimes (\bar{x}'\bar{y}')\big)$ (indices modulo 3),
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = q'(x', y')\theta^i(t_{x,y}) + q(x, y)\theta'^i(t'_{x',y'})$,

  where
  $t_{x,y} = \big(q(x, .)y - q(y, .)x, \frac{1}{2}q(x, y)1 - R_{\bar{x}}R_y, \frac{1}{2}q(x, y)1 - L_{\bar{x}}L_y\big)$

(Vinberg, Allison-Faulkner, Barton-Sudbery, Landsberg-Manivel)

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dim C'

	$\mathfrak{g}(C, C')$	1	2	4	8
dim C	1	A_1	$\tilde{A}_2$	C_3	F_4
	2	$\tilde{A}_2$	$\tilde{A}_2 \oplus \tilde{A}_2$	$\tilde{A}_5$	$\tilde{E}_6$
	4	C_3	$\tilde{A}_5$	D_6	E_7
	8	F_4	$\tilde{E}_6$	E_7	E_8

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- $\tilde{A}_2$ denotes a form of $\mathfrak{pgl}_3$, so $[\tilde{A}_2, \tilde{A}_2]$ is a form of $\mathfrak{psl}_3$.
- $\tilde{A}_5$ denotes a form of $\mathfrak{pgl}_6$, so $[\tilde{A}_5, \tilde{A}_5]$ is a form of $\mathfrak{psl}_6$.
- $\tilde{E}_6$ is not simple, but $[\tilde{E}_6, \tilde{E}_6]$ is a codimension 1 simple ideal.

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Third row dim $C = 4$, so $C = Q$ is a quaternion algebra and

$$\begin{aligned}\mathcal{T}(C,J) &= \mathrm{ad}_{Q_0} \oplus (Q_0 \otimes J_0) \oplus \mathfrak{der}\, J \\ &\simeq (Q_0 \otimes J) \oplus \mathfrak{der}\, J.\end{aligned}$$

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Up to now, everything works for arbitrary Jordan algebras in characteristic $\neq 2$, and even for Jordan superalgebras.

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$\dim C = 8$. If the characteristic is $\neq 2, 3$, then $\mathfrak{Der}\, C = \mathfrak{g}_2$ is simple of type G_2, C_0 is its smallest nontrivial irreducible module, and

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is a G_2-*graded* Lie algebra. Essentially, all the G_2-graded Lie algebras appear in this way [Benkart-Zelmanov 1996].

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- $\mathcal{T}(C, J(V)) \simeq G(3) \quad (\dim V = 2,\ V = V_{\bar{1}})$.
- $\mathcal{T}(C, D_2) \simeq F(4)$.
- $\mathcal{T}(C, K_{10})$ in characteristic 5!!
  This is a new simple modular Lie superalgebra, whose even part is $\mathfrak{so}_{11}$ and odd part its spin module.

A supermagic rectangle

$\mathcal{T}(C, J)$	$H_3(k)$	$H_3(k \times k)$	$H_3(\mathrm{Mat}_2(k))$	$H_3(C(k))$	$J(V)$	D_t	K_{10}
k	A_1	A_2	C_3	F_4	A_1	$B(0,1)$	$B(0,1) \oplus B(0,1)$
$k \times k$	A_2	$A_2 \oplus A_2$	A_5	E_6	$B(0,1)$	$A(1,0)$	$C(3)$
$\mathrm{Mat}_2(k)$	C_3	A_5	D_6	E_7	$B(1,1)$	$D(2,1;t)$	$F(4)$
$C(k)$	F_4	E_6	E_7	E_8	$G(3)$	$F(4)$ $(t=2)$	$\mathcal{T}(C(k), K_{10})$ (char 5)

A supermagic rectangle: the new columns

$\mathcal{T}(C,J)$	$J(V)$	D_t	K_{10}
k	A_1	$B(0,1)$	$B(0,1) \oplus B(0,1)$
$k \times k$	$B(0,1)$	$A(1,0)$	$C(3)$
$\mathrm{Mat}_2(k)$	$B(1,1)$	$D(2,1;t)$	$F(4)$
$C(k)$	$G(3)$	$F(4)$ $(t=2)$	$\mathcal{T}(C(k),K_{10})$ (char 5)

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If the characteristic is 3 and $\dim C = 8$, then $\mathfrak{der}\, C$ is no longer simple, but contains the simple ideal $\operatorname{ad} C_0$ (a form of $\mathfrak{psl}_3$). It makes sense to consider:

$$\begin{aligned}\tilde{\mathcal{T}}(C, J) &= \operatorname{ad} C_0 \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J \\ &\simeq (C_0 \otimes J) \oplus \mathfrak{der}\, J.\end{aligned}$$

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$\tilde{\mathcal{T}}(C, J)$ becomes a Lie algebra if and only if J is a commutative and alternative algebra (these conditions imply the Jordan identity).

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The simple commutative alternative algebras are just the fields, so nothing interesting appears here.

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(i) fields,

(ii) $J(V)$, the Jordan superalgebra of a superform on a two dimensional odd space V,

(iii) $B = B(\Gamma, D, 0) = \Gamma \oplus \Gamma u$, where

- Γ is a commutative associative algebra,
- $D \in \mathfrak{Der}\,\Gamma$ such that Γ is D-simple,
- $a(bu) = (ab)u = (au)b, \quad (au)(bu) = aD(b) - D(a)b,$ $\forall a, b \in \Gamma.$

Fourth "superrow", characteristic 3

Example (Divided powers)

$\Gamma = \mathcal{O}(1;n) = \operatorname{span}\left\{t^{(r)} : 0 \leq r \leq 3^n - 1\right\}$,

$$t^{(r)}t^{(s)} = \binom{r+s}{r}t^{(r+s)},$$

$$D(t^{(r)}) = t^{(r-1)}.$$

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Both simple Lie superalgebras have been considered in a completely different way by Bouarroudj and Leites (2006).

Composition superalgebras

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Definition

A superalgebra $C = C_{\bar{0}} \oplus C_{\bar{1}}$, endowed with a regular quadratic superform $q = (q_{\bar{0}}, b)$, called the *norm*, is said to be a *composition superalgebra* in case

$$q_{\bar{0}}(x_{\bar{0}}y_{\bar{0}}) = q_{\bar{0}}(x_{\bar{0}})q_{\bar{0}}(y_{\bar{0}}),$$

$$b(x_{\bar{0}}y, x_{\bar{0}}z) = q_{\bar{0}}(x_{\bar{0}})b(y, z) = b(yx_{\bar{0}}, zx_{\bar{0}}),$$

$$b(xy, zt) + (-1)^{|x||y|+|x||z|+|y||z|}b(zy, xt) = (-1)^{|y||z|}b(x, z)b(y, t),$$

The unital composition superalgebras are termed *Hurwitz superalgebras*.

Composition superalgebras: examples (Shestakov)

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$$B(1,2) = k1 \oplus V,$$

char $k = 3$, V a two dim'l vector space with a nonzero alternating bilinear form $\langle .|.\rangle$, with

$$1x = x1 = x, \quad uv = \langle u|v\rangle 1, \qquad q_{\bar{0}}(1) = 1, \quad b(u,v) = \langle u|v\rangle,$$

is a Hurwitz superalgebra. (As a superalgebra, this is just our previous $J(V)$.)

Composition superalgebras: examples (Shestakov)

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$$B(4,2) = \mathrm{End}_k(V) \oplus V,$$

k and V as before, $\mathrm{End}_k(V)$ is equipped with the symplectic involution $f \mapsto \bar{f}$, $(\langle f(u)|v\rangle = \langle u|\bar{f}(v)\rangle)$, the multiplication is given by:

- the usual multiplication (composition of maps) in $\mathrm{End}_k(V)$,
- $v \cdot f = f(v) = \bar{f} \cdot v$ for any $f \in \mathrm{End}_k(V)$ and $v \in V$,
- $u \cdot v = \langle .|u\rangle v \ \big(w \mapsto \langle w|u\rangle v\big) \in \mathrm{End}_k(V)$ for any $u, v \in V$,

and with quadratic superform

$$q_{\bar{0}}(f) = \det f, \quad b(u,v) = \langle u|v\rangle,$$

is a Hurwitz superalgebra.

Composition superalgebras: classification

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Theorem (E.-Okubo 2002)

Any unital composition superalgebra is either:

- *a Hurwitz algebra,*
- *a $\mathbb{Z}_2$-graded Hurwitz algebra in characteristic 2,*
- *isomorphic to either $B(1,2)$ or $B(4,2)$ in characteristic 3.*

Composition superalgebras: classification

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- *a $\mathbb{Z}_2$-graded Hurwitz algebra in characteristic* 2,
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Hurwitz superalgebras can be plugged into the symmetric construction of Freudenthal-Tits Magic Square $\mathfrak{g}(C, C')$.

Supermagic Square (char 3, Cunha-E. 2007)

$\mathfrak{g}(C, C')$	k	$k \times k$	$\mathrm{Mat}_2(k)$	$C(k)$	$B(1,2)$	$B(4,2)$
k	A_1	$\tilde{A}_2$	C_3	F_4	$6 \vert 8$	$21 \vert 14$
$k \times k$		$\tilde{A}_2 \oplus \tilde{A}_2$	$\tilde{A}_5$	$\tilde{E}_6$	$11 \vert 14$	$35 \vert 20$
$\mathrm{Mat}_2(k)$			D_6	E_7	$24 \vert 26$	$66 \vert 32$
$C(k)$				E_8	$55 \vert 50$	$133 \vert 56$
$B(1,2)$					$21 \vert 16$	$36 \vert 40$
$B(4,2)$						$78 \vert 64$

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$B(1,2)$					$21\|16$	$36\|40$
$B(4,2)$						$78\|64$

Notation: $\mathfrak{g}(n, m)$ will denote the superalgebra $\mathfrak{g}(C, C')$, with $\dim C = n$, $\dim C' = m$.

Lie superalgebras in the Supermagic Square

	$B(1,2)$	$B(4,2)$
k	$\mathfrak{psl}_{2,2}$	$\mathfrak{sp}_6 \oplus (14)$
$k \times k$	$\big(\mathfrak{sl}_2 \oplus \mathfrak{pgl}_3\big) \oplus \big((2) \otimes \mathfrak{psl}_3\big)$	$\mathfrak{pgl}_6 \oplus (20)$
$\mathrm{Mat}_2(k)$	$\big(\mathfrak{sl}_2 \oplus \mathfrak{sp}_6\big) \oplus \big((2) \otimes (13)\big)$	$\mathfrak{so}_{12} \oplus spin_{12}$
$C(k)$	$\big(\mathfrak{sl}_2 \oplus \mathfrak{f}_4\big) \oplus \big((2) \otimes (25)\big)$	$\mathfrak{e}_7 \oplus (56)$
$B(1,2)$	$\mathfrak{so}_7 \oplus 2spin_7$	$\mathfrak{sp}_8 \oplus (40)$
$B(4,2)$	$\mathfrak{sp}_8 \oplus (40)$	$\mathfrak{so}_{13} \oplus spin_{13}$

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The simple Lie superalgebra $\mathfrak{g}(2,3)' = [\mathfrak{g}(2,3), \mathfrak{g}(2,3)]$ is isomorphic to our previous $\tilde{\mathcal{T}}(C(k), J(V))$.

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$$r = 1,2,4,8, \qquad T = \begin{pmatrix} k & J \\ J & k \end{pmatrix}, \quad J = H_3(C).$$

The Supermagic Square and Jordan superalgebras

	$B(1,2)$	$B(4,2)$
k	$\mathfrak{der}\big(H_3(B(1,2))\big)$	$\mathfrak{der}\big(H_3(B(4,2))\big)$
$k \times k$	$\mathfrak{pstr}\big(H_3(B(1,2))\big)$	$\mathfrak{pstr}\big(H_3(B(4,2))\big)$
$\mathrm{Mat}_2(k)$	$\mathcal{TKK}\big(H_3(B(1,2))\big)$	$\mathcal{TKK}\big(H_3(B(4,2))\big)$
$C(k)$		
$B(1,2)$	$\mathcal{TKK}(K_9)$	
$B(4,2)$		

Simple modular Lie superalgebras with a Cartan matrix

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The finite dimensional modular Lie superalgebras with indecomposable symmetrizable Cartan matrices (or *contragredient Lie superalgebras*) over algebraically closed fields have been classified by Bouarroudj, Grozman and Leites (2009), under some extra technical hypotheses.

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For characteristic $p \geq 3$, apart from the Lie superalgebras obtained as the analogues of the Lie superalgebras in the classification in characteristic 0, by reducing the Cartan matrices modulo p, there are only the following exceptions:

Simple modular Lie superalgebras with a Cartan matrix

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The superalgebra $\mathfrak{el}(5;5)$ is the Lie superalgebra $\mathcal{T}(C(k), K_{10})$ considered previously.

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The superalgebra $\mathfrak{el}(5;3)$ lives (as a natural maximal subalgebra) in the Lie superalgebra $\mathfrak{g}(3,8)$ of the Supermagic Square as follows:

- $\mathfrak{el}(5;3)_{\bar{0}} = \mathfrak{sl}_2 \oplus \mathfrak{so}_9 \leq \mathfrak{sl}_2 \oplus \mathfrak{f}_4 = \mathfrak{g}(3,8)_{\bar{0}}$,
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- $\mathfrak{el}(5;3)_{\bar{1}} = (2) \otimes (C(k) \oplus C(k)) \leq (2) \otimes \hat{J} = \mathfrak{g}(3,8)_{\bar{1}}$,
  $\Big(\hat{J} = J_0/k1$ contains three copies of $C(k)$ in the off-diagonal entries.$\Big)$

Freudenthal-Tits Magic Square

A supermagic rectangle

A supermagic square

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char. 5: In characteristic 5 one can add the new simple Lie superalgebra (without counterpart in Kac's classification) $\mathfrak{el}(5; 5) = \mathcal{T}(C(k), K_{10})$.

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- By using a symmetric construction in terms of two Hurwitz algebras, and extending it (only in characteristic 3) with the use of composition superalgebras.

  Ten new simple Lie superalgebras are obtained: $\mathfrak{g}(r,3)'$ $(r = 2,4,8)$, $\mathfrak{g}(r,6)'$ $(r = 1,2,4,8)$, $\mathfrak{g}(3,3)$, $\mathfrak{g}(3,6)$ and $\mathfrak{g}(6,6)$.

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That's all. Thanks